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TIME-LIMITED MODEL REDUCTION FOR SEMI-MARKOVIAN JUMP SYSTEMS BASED ON GENERALIZED GRAMIANS

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Abstract. *This paper is concerned with the approximation problem for semi-Markovian jump systems (SMJSs) based on generalized time-limited Gramians. For a given SMJS with partially known transition probabilities, the generalized time-limited Gramians are defined and then solved by using time-interval input/output energy functions. The proposed model reduction algorithm can solve the synchronization jump problem, and also achieve the required reduced accuracy within the expected time-interval. Specially, the reduced error of the proposed algorithm is lower compared with the balanced truncation method, which has more theoretical value in time-interval control synthesis. Finally, a numeral example including four subsystems is given to estimate the effectiveness of the proposed results.*

Keywords: Semi-Markovian jump systems (SMJSs), Time-limited Gramians, Partially assess transition probabilities, Model reduction, Balanced truncation

1. **Introduction.** In the actual industrial process, especially for the intelligent transportation, smart power grid and aerospace system, etc., it will often face the challenges of complex environment, multi-task coordination, demand of high precision and high performance. Stochastic switched system is a special class of hybrid systems, which can effectively model the system mutation caused by such as environmental mutation, component failure, and even human factors during the normal operation [1, 2]. Therefore, the research has been widely concerned in the field of control. In recent years, a wealth of research achievements have been made in stochastic switched systems, such as system stabilization [3], filter design [4], fault diagnosis [5], output feedback control [6] and sliding mode control [7, 8, 9]. However, most of these results focus on the analysis and synthesis of Markov jump systems. There are still many difficulties that need to be solved, such as asynchronous jump and inhomogeneous phenomena.

Semi-Markovian jump systems (SMJSs) have a wider practical application value, for SMJSs relax the assumption that the sojourn time is memoryless [10, 11, 12, 13]. For an SMJS whose sojourn time belongs to Weibull distribution, Shen et al. [14] proposed the reliable mixed passive/H_∞ filter design scheme under sensor failures. For an SMJS whose sojourn time obeys phase type distribution, Wang and Zhu [15] proposed a stochastic stability conditions based on the multiple Lyapunov functions. And the results are extended

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to the almost deterministic exponential stability analysis of the SMJS, whose sojourn time is not subject to a particular form of probability distribution. It is not difficult to see that, the probability distribution that sojourn time obeys is different and the analysis method adopted is different, however, the general principle is to solve the multimodal problem of stochastic jump systems and strong coupling problem of Lyapunov parameters.

In recent years, advanced techniques such as linear matrix inequality, cone complementarity linearization, convex linear optimization algorithm, projection theorem and matrix equilibrium transformation, have been applied to solving the model reduction problem of linear time-invariant systems. And effective model reduction methods have been obtained, such as H_2 model reduction method, aggregation method [16], Hankel norm optimization [17], moment matching method [18, 19], balanced truncation method [20] and H_∞ model reduction method [21]. The balanced truncation method is simple and feasible for it can maintain the main performance of the original system. It has been successfully applied to solving the model reduction problem of stochastic jump systems, such as non-minimum phase system, switched system [22] and Markov jump system [23], and the reduced-order error is limited within the H_2 norm. At present, most of the achievements are in the infinite time domain, which means, the system approximates the original system in the time period $[0, +\infty)$. However, in practical applications, for example, the problem of finite time optimization control, the approximation problem often needs to be solved within the finite time $[t_1, t_2]$. Redmann et al. [24, 25, 26] proposed the model reduction method for linear time-invariant systems based on H_2 optimization algorithm and balanced truncation algorithm in finite time interval respectively, and verified the bound of the reduced-order error in balanced truncation algorithm. Furthermore, the time-limited model reduction algorithm based on balanced truncation is extended to continuous-time large systems [27], discrete-time fractional-order systems [28] and positive-real systems [29], respectively. And it is verified that the reduced-order system can achieve the required precision in the expected time range. However, the time-limited model reduction problem for SMJSs with partially assess transition probabilities remains unsolved.

In view of this, this paper investigates the time-limited model reduction algorithm for a special class of SMJSs with partially known transition probabilities based on generalized Gramians within time-interval $[t_1, t_2]$. The main innovations are given as follows:

1) settle the model reduction problem for a special class of SMJSs with partially known transition probabilities within the time-interval $[t_1, t_2]$;
2) verify that the reduced-order model could jump with the original system synchronously;
3) maintain the main structure, main input/output performances of the original system, and the reduced-order error has an upper bound.

The rest of this paper is organized as follows. The problem statement and preliminaries are given in Section 2. The main results including the solving methods of time-limited Gramians, model reduction algorithm on time-interval $[t_1, t_2]$ for SMJSs and the corresponding reduction error are presented in Section 3. In Section 4, a numerical example including four subsystems is introduced to estimate the proposed results and Section 5 concludes this paper.

2. **Problem Statement and Preliminaries.**

Consider the following continuous-time stochastic systems with semi-Markov parameters over a probability space $(\mho, \mathcal{F}, P_r)$:

$$(\Sigma): \quad \begin{cases} \dot{x}(t) = A(\eta_t)x(t) + B(\eta_t)u(t), \\ y(t) = C(\eta_t)x(t) + D(\eta_t)u(t), \ x(0) = x_0, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$ are the state vector and measured output vector, respectivily; and $u(t) \in \mathbb{R}^m$ is the control input which belongs to $\mathcal{L}_2[0, \infty)$. Let $\{\eta_t, t \geq 0\}$ be a semi-Markov chain taking values in the state space $\mathcal{N} = \{1, 2, \ldots, N\}$ and assuming that the time-varying transition probabilities satisfies:

$$\Pr(\eta_{t+h} = j|\eta_t = i) = \begin{cases} \alpha_{ij}(h)h + o(h), & \text{if } i \neq j, \\ 1 + \alpha_{ij}(h)h + o(h), & \text{if } i = j, \end{cases}$$

where h $(h > 0)$ is called the sojourn time and $o(h)$ is the little-o notation, satisfying $\lim_{h\to 0} \frac{o(h)}{h} = 0$; $\alpha_{ij}(h)$ is the sojourn-time-based transition probability rate from subsystem i at time t to subsystem j at time $t+h$, and satisfies $\alpha_{ij}(h) \geq 0$ $(i, j \in \mathcal{N}, i \neq j)$ and $\alpha_{ii}(h) = -\sum_{j=1,j\neq i}^{N} \alpha_{ij}(h)$. Motivated by [14], we assume transition probability rate is bounded as $\underline{\alpha}_{ij} \leq \alpha_{ij}(h) \leq \bar{\alpha}_{ij}$ with $\underline{\alpha}_{ij}$ and $\bar{\alpha}_{ij}$ being constant scalars. Here, we assume that

$$\alpha_{ij}(h) = \alpha_{ij} + \Delta\alpha_{ij},\ \alpha_{ij} = \frac{1}{2}(\bar{\alpha}_{ij} + \underline{\alpha}_{ij}),\ |\Delta\alpha_{ij}| \leq \kappa_{ij} = 0.5(\bar{\alpha}_{ij} - \underline{\alpha}_{ij}).$$

Furthermore, time-varying transition probabilities are partially unknown, and we define

$$\mathcal{N} = \mathcal{N}^i_{UK} \cup \mathcal{N}^i_{UC},\ \mathcal{N}^i_{UK} \triangleq \{j : \alpha_{ij}(h) \text{ is unknown}\},\ \mathcal{N}^i_{UC} \triangleq \{j : \alpha_{ij}(h) \text{ is uncertain}\}. \quad (2)$$

In order to obtain the expected research results, the following definitions and lemmas for the above mathematical models are adopted here.

Assumption 2.1. *The considered SMJS in (1) is assumed to be asymptotically stable and minimal.*

Definition 2.1. *The considered SMJS in (1) with $u(t) = 0$ is said to be asymptotically stable if*

$$\mathbb{E}\left\{\int_0^\infty \|x(t)\|^2 dt|x_0, \eta_0\right\} < \infty,$$

holds for any initial vector $x_0 \in \mathbb{R}^n$, $\eta_0 \in \mathcal{N}$.

Lemma 2.1. *The SMJS in (1) is asymptotically stable, if there exists a set of matrices $Q_i = Q_i^T > 0$ satisfying*

$$A_i^T Q_i + Q_i A_i + \sum_{j=1}^{N} \alpha_{ij}(h) Q_j < 0,\ i \in \mathcal{N}.$$

Or dually there exist matrices $P_i = P_i^T > 0$ satisfying

$$A_i P_i + P_i A_i^T + \sum_{j=1}^{N} \alpha_{ij}(h) P_j < 0,\ i \in \mathcal{N}.$$

The following lemma is used to deal with the time-varying transition probability rates for the considered SMJS.

Lemma 2.2. *For any given scalar ϵ and matrix $\Gamma \in \mathbb{R}^{n\times n}$, the following inequality holds*

$$\epsilon\left(\Gamma + \Gamma^T\right) \leq \epsilon^2 X + \Gamma X^{-1}\Gamma^T,$$

for any positive definite symmetric matrix $X \in \mathbb{R}^{n\times n}$.

Definition 2.2. *Consider the SMJS in (1), the time-limited Gramians within time-interval $[t_1, t_2]$ are defined as*

$$\mathcal{P}_i = \int_{t_1}^{t_2} e^{A_i\tau} B_i B_i^T e^{A_i^T\tau} d\tau,\ \mathcal{Q}_i = \int_{t_1}^{t_2} e^{A_i^T\tau} C_i^T C_i e^{A_i\tau} d\tau,$$

satisfying

$$\mathcal{P}_i = \hat{P}_i(t_1) - \hat{P}_i(t_2), \; \hat{P}_i(t) = P_i - e^{A_i t} P_i e^{A_i^T t},$$
$$\mathcal{Q}_i = \hat{Q}_i(t_1) - \hat{Q}_i(t_2), \; \hat{Q}_i(t) = Q_i - e^{A_i^T t} Q_i e^{A_i t},$$

where $P_i \triangleq \int_0^\infty e^{A_i\tau} B_i B_i^T e^{A_i^T \tau} d\tau$ *and* $Q_i \triangleq \int_0^\infty e^{A_i^T\tau} C_i^T C_i e^{A_i \tau} d\tau$ $(i \in \mathcal{N})$.

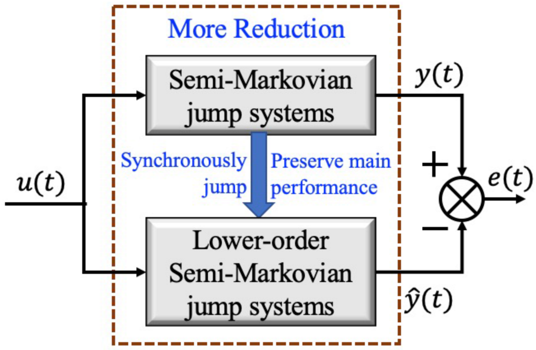

FIGURE 1. The basic framework of model reduction for SMJSs

As shown in Figure 1, this paper aims to find a lower-order system with the same structure:

$$(\hat{\Sigma}) : \quad \begin{cases} \dot{\hat{x}}(t) = \hat{A}(\eta_t)\hat{x}(t) + \hat{B}(\eta_t)u(t), \\ \hat{y}(t) = \hat{C}(\eta_t)\hat{x}(t) + \hat{D}(\eta_t)u(t), \end{cases} \tag{3}$$

where $\hat{x}(t) \in \mathbb{R}^k$ with $1 \leq k < n$. Matrices $\hat{A}(\eta_t)$, $\hat{B}(\eta_t)$, $\hat{C}(\eta_t)$ and $\hat{D}(\eta_t)$, to be determined later, can jump synchronously with the original system. In order to evaluate the precision of the proposed time-limited model reduction algorithm, the following error system is employed:

$$(\Sigma_e) : \quad \begin{cases} \dot{x}_e(t) = A_e(\eta_t)x_e(t) + B_e(\eta_t)u(t), \\ e(t) = C_e(\eta_t)x_e(t) + D_e(\eta_t)u(t), \end{cases} \tag{4}$$

where $x_e(t) \triangleq [x^T(t) \quad \hat{x}^T(t)]^T$, $e(t) \triangleq y(t) - \hat{y}(t)$, and

$$A_e(\eta_t) \triangleq \begin{bmatrix} A(\eta_t) & 0 \\ 0 & \hat{A}(\eta_t) \end{bmatrix}, \; B_e(\eta_t) \triangleq \begin{bmatrix} B(\eta_t) \\ \hat{B}(\eta_t) \end{bmatrix},$$
$$C_e(\eta_t) \triangleq [\, C(\eta_t) \quad -\hat{C}(\eta_t) \,], \; D_e(\eta_t) \triangleq D(\eta_t) - \hat{D}(\eta_t).$$

Definition 2.3. *Given a given scalar* $\gamma > 0$, *the error system in (4) is asymptotically stable and has an* H_∞ *performance index* γ *in time-interval* $[t_1, t_2]$ *if the system is stable and the following inequality holds under zero initial condition (i.e.,* $x(t) = 0$, $t \leq 0$*)*

$$\mathbb{E}\left\{\int_{t_1}^{t_2} \|e(t)\|^2 dt\right\} < \gamma^2 \int_{t_1}^{t_2} \|u(t)\|^2 dt,$$

for all non-zero $u(t) \in \mathcal{L}_2[0, \infty)$.

3. Main Results.

3.1. The time-limited Gramians for SMJSs.

A newly balanced order reduction algorithm based on time-limited Gramians is proposed in this paper, which aims to solve the model order reduction problem of SMJSs. Firstly, the time-limited Gramians solution algorithm is given in the form of theorems.

Theorem 3.1. *Consider the continuous-time SMJS in (1), the time-limited controllability Gramians $\mathcal{P}_i$ in time-interval $[t_1, t_2]$ can be obtained, if exist matrices W_i and X_{ij} ($i, j \in \mathcal{N}$) satisfying:*

$$\begin{bmatrix} A_i\mathcal{P}_i + \mathcal{P}_iA_i^T + \mathcal{X}_{\mathcal{P}_i} + \sum\limits_{j\in\mathcal{N}_{UC}}^{N} \alpha_{ij}(\mathcal{P}_j - W_i) + \sum\limits_{j\in\mathcal{N}_{UC}, j\neq i}^{N} \frac{\kappa_{ij}^2}{4}X_{ij} & \mathcal{Z}_{\mathcal{P}_i} \\ * & -\Lambda_{\mathcal{P}_i} \end{bmatrix} < 0, \ j \in \mathcal{N}_{UC} \quad (5)$$

$$\mathcal{P}_j - W_i < 0, \ j \in \mathcal{N}_{UK}, \ i \neq j, \quad (6)$$

$$\mathcal{P}_j - W_i > 0, \ j \in \mathcal{N}_{UK}, \ i = j, \quad (7)$$

where

$$\begin{cases} \mathcal{X}_{\mathcal{P}_i} \triangleq e^{A_it_1}B_iB_i^Te^{A_i^Tt_1} - e^{A_it_2}B_iB_i^Te^{A_i^Tt_2}, \\ \mathcal{Z}_{\mathcal{P}_i} \triangleq [\mathcal{P}_i - \mathcal{P}_1 \quad \cdots \quad \mathcal{P}_i - \mathcal{P}_{i-1} \quad \mathcal{P}_i - \mathcal{P}_{i+1} \quad \cdots \quad \mathcal{P}_i - \mathcal{P}_N], \\ \Lambda_{\mathcal{P}_i} \triangleq \operatorname{diag}\{X_{i1}, \ldots, X_{i(i-1)}, X_{i(i+1)}, \ldots, X_{iN}\}, \end{cases}$$

then the SMJS is stochastically stable, and the energy required to drive the states from $x(t_1, x_{t_1}, u, \eta_{t_1}) = x_{t_1}$ to $x(t_2, x_{t_2}, u, \eta_{t_2}) = x_{t_2}$ is bounded:

$$\mathbb{E}\left\{\int_{t_1}^{t_2} x_t^T\mathcal{P}_i^{-1}x_tdt\right\} < \inf_{u\in\mathcal{L}_2[t_1,t_2]} \int_{t_1}^{t_2} \|u(t)\|^2dt, \quad (8)$$

for all non-zero $u(t) \in \mathcal{L}_2[t_1, t_2]$.

Proof: Firstly, the SMJS in (1) is asymptotically stable using Lemma 2.1 and (5)-(7). Suppose that the state of the SMJS is driven from $x(t_1)$ to $x(t_2)$ with the following external input $u(t) : [t_1, t_2] \to \mathbb{R}^m$:

$$u(t) \triangleq B_i^Te^{A_i^Tt}\mathcal{P}_i^{-1}x_t,$$

then the energy that needs to be input is

$$\int_{t_1}^{t_2} \|u(t)\|^2dt = \int_{t_1}^{t_2} x_t^T\mathcal{P}_i^{-1}e^{A_it}B_iB_i^Te^{A_i^Tt}\mathcal{P}_i^{-1}x_tdt = \int_{t_1}^{t_2} x_t^T\mathcal{P}_i^{-1}x_tdt,$$

therefore, (8) is proved.

Furthermore, the following parameter-based Lyapunov function is considered:

$$V(x, t, \eta_t) = x^T(t)\mathcal{P}^{-1}(\eta_t)x(t),$$

and $\mathcal{P}$ is positive definite symmetric matrix. From (8), we can obtain that

$$\mathcal{L}V(x, t, i) - u^T(t)u(t) = 2\dot{x}^T(t)\mathcal{P}_i^{-1}x(t) + x^T(t)\sum_{j=1}^{N}\left[\alpha_{ij}(h)\mathcal{P}_j^{-1}\right]x(t) - u^T(t)u(t),$$

where $\xi(t) \triangleq [x(t)^T \quad u(t)^T]^T$. Multiplying matrix $\mathcal{P}_i$ on both sides of above, we can obtain that

$$\begin{aligned} &A_i\mathcal{P}_i + \mathcal{P}_iA_i^T + \sum_{j=1}^{N}\alpha_{ij}(h)\mathcal{P}_j \\ &= \int_{t_1}^{t_2}\left(A_ie^{A_i\tau}B_iB_i^Te^{A_i^T\tau} + e^{A_i\tau}B_iB_i^Te^{A_i^T\tau}A_i^T\right)d\tau + \sum_{j=1}^{N}\alpha_{ij}(h)\mathcal{P}_j \\ &= \int_{t_1}^{t_2} d\left(e^{A_i\tau}B_iB_i^Te^{A_i^T\tau}\right) = -\mathcal{X}_{\mathcal{P}_i}. \end{aligned} \quad (9)$$

In addition, the transition probabilities considered in this paper are partially known and satisfy that $\sum_{j=1}^{N}\alpha_{ij}(h)=0$. Then (9) can be obtained that

$$\begin{aligned} & A_i\mathcal{P}_i+\mathcal{P}_iA_i^T+\mathcal{X}_{\mathcal{P}_i}+\sum_{j=1}^{N}\alpha_{ij}(h)\mathcal{P}_j-\sum_{j=1}^{N}\alpha_{ij}(h)W_i \\ = {} & A_i\mathcal{P}_i+\mathcal{P}_iA_i^T+\mathcal{X}_{\mathcal{P}_i}+\sum_{j\in\mathcal{N}_{UC}}^{N}\alpha_{ij}(h)(\mathcal{P}_j-W_i)+\sum_{j\in\mathcal{N}_{UK}}^{N}\alpha_{ij}(h)(\mathcal{P}_j-W_i), \end{aligned}$$

where the transition probability $\alpha_{ij}(h)$ depends on the sojourn time h, which means that the above equation contains infinite number of equations. To solve thus kind of problem, we assume that the time-varying transition probability $\alpha_{ij}(h)$ satisfies that

$$\alpha_{ij}(h)=\alpha_{ij}+\Delta\alpha_{ij},\quad \alpha_{ij}=\frac{1}{2}(\bar{\alpha}_{ij}+\underline{\alpha}_{ij}),\quad |\alpha_{ij}|\le\kappa_{ij}=\frac{1}{2}(\bar{\alpha}_{ij}-\underline{\alpha}_{ij}).$$

Using Lemma 2.2, we have

$$\begin{aligned} & A_i\mathcal{P}_i+\mathcal{P}_iA_i^T+\mathcal{X}_{\mathcal{P}_i}+\sum_{j\in\mathcal{N}_{UC}}^{N}\alpha_{ij}(h)(\mathcal{P}_j-W_i) \\ & +\sum_{j\in\mathcal{N}_{UC},j\neq i}^{N}\left[\frac{1}{2}\Delta\alpha_{ij}(\mathcal{P}_j-\mathcal{P}_i)+\frac{1}{2}\Delta\alpha_{ij}(\mathcal{P}_j-\mathcal{P}_i)\right] \\ \le {} & A_i\mathcal{P}_i+\mathcal{P}_iA_i^T+\mathcal{X}_{\mathcal{P}_i}+\sum_{j\in\mathcal{N}_{UC}}^{N}\alpha_{ij}(h)(\mathcal{P}_j-W_j) \\ & +\sum_{j\in\mathcal{N}_{UC},j\neq i}^{N}\left[\frac{\kappa_{ij}^2}{4}X_{ij}+(\mathcal{P}_j-\mathcal{P}_i)X_{ij}^{-1}(\mathcal{P}_j-\mathcal{P}_i)\right]. \end{aligned}$$

Applying Schur complement lemma, the above inequality is equivalent to (5)-(7). Thus the proof is completed. $\square$

Theorem 3.2. *Consider the continuous-time SMJS in (1), the time-limited observability Gramians $\mathcal{Q}_i$ in time interval $\mathcal{T}=[t_1,t_2]$ can be obtained, if there exist matrices V_i and Y_{ij} $(i,j\in\mathcal{N})$ satisfying:*

$$\begin{bmatrix} A_i^T\mathcal{Q}_i+\mathcal{Q}_iA_i+\mathcal{X}_{\mathcal{Q}_i}+\sum\limits_{j\in\mathcal{N}_{UC}}^{N}\alpha_{ij}(\mathcal{Q}_j-V_i)+\sum\limits_{j\in\mathcal{N}_{UC},j\neq i}^{N}\frac{\kappa_{ij}^2}{4}Y_{ij} & \mathcal{Z}_{\mathcal{Q}_i} \\ * & -\Lambda_{\mathcal{Q}_i}\end{bmatrix}<0,\ j\in\mathcal{N}_{UC} \tag{10}$$

$$\mathcal{Q}_j-V_i<0,\ j\in\mathcal{N}_{UK},\ i\neq j, \tag{11}$$

$$\mathcal{Q}_j-V_i>0,\ j\in\mathcal{N}_{UK},\ i=j, \tag{12}$$

where

$$\begin{cases} \mathcal{X}_{\mathcal{Q}_i}\triangleq e^{A_i^Tt_1}C_i^TC_ie^{A_it_1}-e^{A_i^Tt_2}C_i^TC_ie^{A_it_2}, \\ Z_{\mathcal{Q}_i}\triangleq[\mathcal{Q}_i-\mathcal{Q}_1\quad\cdots\quad\mathcal{Q}_i-\mathcal{Q}_{i-1}\quad\mathcal{Q}_i-\mathcal{Q}_{i+1}\quad\cdots\quad\mathcal{Q}_i-\mathcal{Q}_N], \\ \Lambda_{\mathcal{Q}_i}\triangleq\operatorname{diag}\{Y_{i1},\ldots,Y_{i(i-1)},Y_{i(i+1)},\ldots,Y_{iN}\}, \end{cases}$$

then the SMJS is stochastically stable, and the average energy of the output is bounded ($u(t)=0$):

$$\mathbb{E}\left\{\int_{t_1}^{t_2}\|y(t,x_{t_1},0,i)\|^2dt\right\}<\mathbb{E}\left\{x_{t_1}^T\mathcal{Q}_ix_{t_1}\right\}. \tag{13}$$

Proof: Theorem 3.1 and Theorem 3.2 are dual, because the space is limited, the process of proof is omitted. □

It is worth to mention that, there are N controllability and observability time-limited Gramians $\mathcal{P}_i$ and $\mathcal{Q}_i$ for the considered SMJS in (1) obtained by Theorems 3.1 and 3.2. The next step of the proposed model reduction is to solve the balanced transition matrix T_e, satisfying

$$T_e \mathcal{P}_i T_e^T = T_e^{-T} \mathcal{Q}_i T_e^{-1} = \Sigma.$$

What should be mentioned is that the above equation is complex and costly. In order to solve thus problem, the optimized results $\mathcal{P}_g$ and $\mathcal{Q}_g$ are introduced by solving the following minimum optimization problem:

$$\begin{array}{l} \min \operatorname{trace}(\mathcal{P}_i \mathcal{Q}_j) \\ \text{ s.t. (5)-(7) and (10)-(12) for all } (i, j \in \mathcal{N}) \end{array} \tag{14}$$

and we define $\mathcal{P}_g$ and $\mathcal{Q}_g$ as the generalized controllability and observability time-limited Gramians.

3.2. **Model reduction algorithm on time-interval $[t_1, t_2]$.** This paper uses the characteristic that, "equivalent transformation can only change the parameter matrices of the system, cannot change the input/output performance of the original system". In the time-interval $[t_1, t_2]$, the original system is transformed into a balanced form according to the controllability and observability, that is, the state with weaker controllability is also weak observability [30]. If the states with weaker controllability and observability are truncated, the lower-order system will be obtained. The specific model reduction process is shown in Table 1.

3.3. **Reduction error.** Next we will discuss that, there exists an upper bound of the reduced-order error between the original system and the reduced system.

Theorem 3.3. *For a given stable and minimal SMJS in (1), if the generalized Gramians in time-interval* $[t_1, t_2]$ *satisfied that*

$$\mathcal{P}_g = \mathcal{Q}_g = \Sigma = \operatorname{diag}\left\{\Sigma_k, \Sigma_l\right\},$$

where $\Sigma_l = \operatorname{diag}\left\{\sigma_{k+1} I_{l_1}, \sigma_{k+2} I_{l_2}, \ldots, \sigma_n I_{l_s}\right\}$, $\sigma_{k+1} \geq \sigma_{k+2} \geq \cdots \geq \sigma_n > 0$ *and* $l_1 + l_2 + \cdots + l_s = n-k$, *then the lower-order system when truncating the last* $n-k$ *states preserves the stability, and the error between the original SMJS and the obtained lower-order one is bounded within the following* H_∞ *norm*

$$\left\| G_i - \hat{G}_i \right\|_\infty \leq 2 \sum_{j=k+1}^{n} \sigma_j. \tag{18}$$

Proof: First, let $\hat{G}_{l_s i}$ be the realisation of a new reduced-order when truncating the last l_s-th states. The inequality in (18) will be

$$\left\| G_i - \hat{G}_{l_s i} \right\|_\infty \leq 2\sigma_n. \tag{19}$$

According to Definition 2.3, the main purpose is to prove the above error system is asymptotically stable and has an H_∞ performance index $2\sigma_n$ on time interval $[t_1, t_2]$, that is, to find a proper quadratic storage function $V(\cdot) \geq 0$ satisfying

$$\mathbb{E}\left\{\int_{t_1}^{t_2} \|e(t)\|^2 dt\right\} + \mathbb{E}\left\{\int_{t_1}^{t_2} \mathcal{L}V(t) dt + V[t_1] - V[t_2]\right\} \leq 4\sigma_n^2 \int_{t_1}^{t_2} \|u(t)\|^2 dt. \tag{20}$$

TABLE 1. Model reduction algorithm within time interval $[t_1, t_2]$ for SMJSs

Input: State matrices of a stable and minimal SMJS in (1): $\{A_i, B_i, C_i, D_i\}$.
Output: The reduced-order state matrices: $\left\{\hat{A}_i, \hat{B}_i, \hat{C}_i, \hat{D}_i\right\}$.

1. Define an initial condition for the required state matrices $\left\{\hat{A}_{0i}, \hat{B}_{0i}, \hat{C}_{0i}, \hat{D}_{0i}\right\}$.
2. **while** not converged **do**
   (1) Compute the time-limited Gramians using (5)-(7) and (10)-(12) in Theorems 3.1 and 3.2, respectively.
   (2) Use the following optimization algorithm
   $$\begin{aligned} &\min \operatorname{trace}(\mathcal{P}_i \mathcal{Q}_j) \\ &\text{ s.t. (5)-(7) and (10)-(12) for all } (i, j \in \mathcal{N}) \end{aligned}$$
   to solve the generalized time-limited Gramians $\mathcal{P}_g$ and $\mathcal{Q}_g$ within time-interval $[t_1, t_2]$.
   (3) Find the balanced transition matrix T_e, which can convert the original system to a balanced form by using the equivalent transformation, satisfying
   $$T_e \mathcal{P}_g T_e^T = T_e^{-T} \mathcal{Q}_g T_e^{-1} = \Sigma = \operatorname{diag}\{\sigma_1 I_1, \sigma_2 I_2, \ldots, \sigma_n I_n\}, \tag{15}$$
   where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ are the Hankel singular values of the SMJS in (1). It is worth mentioning that, it is hard to solve the balanced transition matrix T_e by using (15) directly and the following algorithm is introduced:
   - Use the Cholesky factorization to decompose the matrix $\mathcal{P}_g$ in time-interval $[t_1, t_2]$: $\mathcal{P}_g = R^T R$;
   - Use the diagonalization to the positive definite symmetric matrix $R\mathcal{Q}_g R^T$: $R\mathcal{Q}_g R^T = U\Sigma^2 U^T$, $U^T U = I$;
   - The balanced transition matrix T_e can be obtained: $T_e = \Sigma^{-1/2} U^T R$.

   (4) The original high-order system is transformed into the following balanced form using the balanced transition matrix T_e obtained above:
   $$\tilde{G}_i = \begin{bmatrix} \tilde{A}_i & \tilde{B}_i \\ \tilde{C}_i & \tilde{D}_i \end{bmatrix} = \begin{bmatrix} T_e A_i T_e^{-1} & T_e B_i \\ C_i T_e^{-1} & D_i \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11i} & \tilde{A}_{12i} & \tilde{B}_{1i} \\ \tilde{A}_{21i} & \tilde{A}_{22i} & \tilde{B}_{2i} \\ \tilde{C}_{1i} & \tilde{C}_{2i} & \tilde{D}_i \end{bmatrix}, \; i \in \mathcal{N} \tag{16}$$
   Now the new state vector is decomposed to
   $$\tilde{x}(t) = \begin{bmatrix} \tilde{x}_1^T(t) & \tilde{x}_2^T(t) \end{bmatrix}^T,$$
   where $\tilde{x}_1(t) \in \mathbb{R}^k$ is the state that should be retained, and $\tilde{x}_2(t) \in \mathbb{R}^{n-k}$ is the state that should be truncated.
   (5) The reduced-order model is
   $$\hat{G}_i(t) = \begin{bmatrix} \hat{A}_i & \hat{B}_i \\ \hat{C}_i & \hat{D}_i \end{bmatrix} = \begin{bmatrix} \hat{A}_{11i} & \hat{B}_{1i} \\ \hat{C}_{1i} & \hat{D}_i \end{bmatrix}. \tag{17}$$
3. **end while**

Before starting, let $E_{l_s} \triangleq \operatorname{diag}\{0, I_{l_s}\}$ and the corresponding state-space realisation of the new reduced-order model be
$$\begin{cases} \dot{\hat{x}}_{l_s}(t) = (I_n - E_{l_s})[A_i \hat{x}_{l_s}(t) + B_i u(t)], \\ \hat{y}_{l_s}(t) = C_i \hat{x}_{l_s}(t) + D_i u(t), \end{cases} \tag{21}$$
where $\hat{x}(t) \in \mathbb{R}^{n-l_s}$ and $\hat{x}_{l_s}(t) = [\hat{x}^T(t) \quad 0]^T$. Now we define that $\epsilon_{l_s}(t) = x(t) - \hat{x}_{l_s}(t)$, $h_{l_s}(t) = x(t) + \hat{x}_{l_s}(t)$ and $\mu_{l_s}(t) = A_i \hat{x}_{l_s} + B_i u(t)$, and the state-space of the corresponding

error system between the original SMJS and the reduced-order one is denoted as

$$\begin{cases} \dot{h}_{l_s}(t) = A_i h_{l_s}(t) + 2B_i u(t) - E_{l_s}\mu_{l_s}(t), \\ \dot{\epsilon}_{l_s}(t) = A_i \epsilon_{l_s}(t) + E_{l_s}\mu_{l_s}(t), \\ e_{l_s}(t) = C_i \epsilon_{l_s}(t). \end{cases} \tag{22}$$

The corresponding inequality in (20) can be expressed as,

$$\mathbb{E}\left\{\int_{t_1}^{t_2}\left\{\|e_{l_s}(t)\|^2 + \mathcal{L}V[x(t),\hat{x}_{l_s}(t)]\right\}dt + V[x,\hat{x}_{l_s},t_1] - V[x,\hat{x}_{l_s},t_2]\right\}$$
$$< 4\sigma_n^2\int_{t_1}^{t_2}\|u(t)\|^2 dt. \tag{23}$$

If we choose the storage function $V(\varpi_s, \delta_s, t, \eta_t = i)$ as:

$$\begin{aligned} V[x(t),\hat{x}_{l_s}(t)] &= \sigma_n^2[x(t)+\hat{x}_{l_s}(t)]^T P_i[x(t)+\hat{x}_{l_s}(t)] + [x(t)-\hat{x}_{l_s}(t)]^T Q_i[x(t)-\hat{x}_{l_s}(t)] \\ &= \sigma_n^2 h_{l_s}^T(t)P_i h_{l_s}(t) + \epsilon_{l_s}^T(t)Q_i\epsilon_{l_s}(t), \end{aligned} \tag{24}$$

where $P_i = P_i^T > 0$, $Q_i = Q_i^T > 0$. The infinitesimal operator $\mathcal{L}$ is considered to be the derivative of the storage function $V[x(t),\hat{x}_{l_s}(t)]$ along the trajectory of the semi-Markov chain, then we can obtain that

$$\begin{aligned} &\mathbb{E}\left\{\int_{t_i}^{t_2}\mathcal{L}V[x(t),\hat{x}_{l_s}(t)]dt + V[x,\hat{x}_{l_s},t_1] - V[x,\hat{x}_{l_s},t_2]\right\} \\ &= \mathbb{E}\Bigg\{\int_{t_i}^{t_2}\Bigg[2\sigma_n^2\dot{h}_{l_s}^T(t)P_i h_{l_s}(t) + 2\dot{\epsilon}_{l_s}^T(t)Q_i\epsilon_{l_s}(t) + \sigma_n^2 h_{l_s}^T(t)\left(\sum_{j=1}^N \alpha_{ij}(h)P_j\right)h_{l_s}(t) \\ &\quad + \epsilon_{l_s}^T(t)\left(\sum_{j=1}^N \alpha_{ij}(h)Q_j\right)\epsilon_{l_s}(t)\Bigg]dt + \sigma_n^2\left[h_{l_s}^T(t_1)P_i h_{l_s}(t_1) - h_{l_s}^T(t_2)P_i h_{l_s}(t_2)\right] \\ &\quad + \epsilon_{l_s}^T(t_1)Q_i\epsilon_{l_s}(t_1) - \epsilon_{l_s}^T(t_2)Q_i\epsilon_{l_s}(t_2)\Bigg\} \\ &= \mathbb{E}\Bigg\{\int_{t_i}^{t_2}\Bigg\{2\sigma_n^2\left[A_i h_{l_s}(t) + 2B_i u(t) - E_{l_s}\mu_{l_s}(t)\right]^T P_i h_{l_s}(t) \\ &\quad + \sigma_n^2 h_{l_s}^T(t)\left(\sum_{j=1}^N \alpha_{ij}(h)P_j\right)h_{l_s}(t) + 2[A_i\epsilon_{l_s}(t) + E_{l_s}\mu_{l_s}(t)]^T Q_i\epsilon_{l_s}(t) \\ &\quad + \epsilon_{l_s}^T(t)\left(\sum_{j=1}^N \alpha_{ij}(h)Q_j\right)\epsilon_{l_s}(t)\Bigg\}dt + \sigma_n^2\left[h_{l_s}^T(t_1)P_i h_{l_s}(t_1) - h_{l_s}^T(t_2)P_i h_{l_s}(t_2)\right] \\ &\quad + \epsilon_{l_s}^T(t_1)Q_i\epsilon_{l_s}(t_1) - \epsilon_{l_s}^T(t_2)Q_i\epsilon_{l_s}(t_2)\Bigg\} \\ &= \mathbb{E}\Bigg\{\int_{t_i}^{t_2}\Bigg\{2\sigma_n^2[A_i h_{l_s}(t) + 2B_i u(t)]^T P_i h_{l_s}(t) + \sigma_n^2 h_{l_s}^T(t)\left(\sum_{j=1}^N \alpha_{ij}(h)P_j\right)h_{l_s}(t) \\ &\quad + 2\epsilon_{l_s}^T(t)A_i^T Q_i\epsilon_{l_s}(t) + \epsilon_{l_s}^T(t)\left(\sum_{j=1}^N \alpha_{ij}(h)Q_j\right)\epsilon_{l_s}(t) \end{aligned}$$

$$+ 2\mu_{l_s}^T(t)E_{l_s}\left[Q_i\epsilon_{l_s}(t) - \sigma_n^2 P_i h_{l_s}(t)\right]\Bigg\}dt + \sigma_n^2\left[h_{l_s}^T(t_1)P_i h_{l_s}(t_1) - h_{l_s}^T(t_2)P_i h_{l_s}(t_2)\right]$$

$$+ \epsilon_{l_s}^T(t_1)Q_i\epsilon_{l_s}(t_1) - \epsilon_{l_s}^T(t_2)Q_i\epsilon_{l_s}(t_2)\Bigg\}$$

$$\leq 4\sigma_n^2\int_{t_i}^{t_2}\|u(t)\|^2dt - \mathbb{E}\left\{\int_{t_i}^{t_2}\epsilon_{l_s}^T(t)C_i^TC_i\epsilon_{l_s}(t)dt\right\}$$

$$+\mathbb{E}\left\{\int_{t_i}^{t_2}2\mu_{l_s}^T(t)E_{l_s}[\epsilon_{l_s}(t) - h_{l_s}(t)]dt\right\},$$

where $2\mu_{l_s}^T(t)E_{l_s}[\epsilon_{l_s}(t) - h_{l_s}(t)] = 0$, that is

$$\mathbb{E}\left\{\int_{t_i}^{t_2}\mathcal{L}V[x(t),\hat{x}_{l_s}(t)]dt + V[t_1] - V[t_2]\right\} < 4\sigma_n^2\int_{t_i}^{t_2}\|u(t)\|^2dt - \mathbb{E}\left\{\int_{t_i}^{t_2}\|e_{l_s}(t)\|^2dt\right\}.$$

Thus relation (20) is satisfied and the first step of the proof is finished.

Similarly, truncating the last l_{s-1} states satisfies

$$\left\|\hat{G}_{l_s i} - \hat{G}_{l_{s-1}i}\right\|_\infty \leq 2\sigma_{n-1},$$

analogy in order,

$$\cdots,\ \left\|\hat{G}_{l_{s-j}i} - \hat{G}_{l_{s-j-1}i}\right\|_\infty \leq 2\sigma_{n-j-1},\ \cdots,\ \left\|\hat{G}_{l_2 i} - \hat{G}_{l_1 i}\right\|_\infty \leq 2\sigma_{k+1}.$$

In conclusion,

$$\left\|G_i - \hat{G}_i\right\|_\infty \leq \left\|G_i - \hat{G}_{l_s i}\right\|_\infty + \cdots + \left\|\hat{G}_{l_{s-j}i} - \hat{G}_{l_{s-j-1}i}\right\|_\infty + \cdots + \left\|\hat{G}_{l_2 i} - \hat{G}_{l_1 i}\right\|_\infty$$

$$= 2\sigma_n + 2\sigma_{n-1} + \cdots + 2\sigma_{n-j-1} + \cdots + 2\sigma_{k+1} = 2\sum_{j=k+1}^{n}\sigma_j,$$

where $\hat{G}_1 = \hat{G}$. Thus the proof is completed. □

4. **Numerical Example.** Consider an SMJS that has four subsystems with the following state matrix parameters:

$$A_1 = \begin{bmatrix} -3.0 & 0.5 & 0.6 & 0.2 \\ 0.0 & -2.5 & 0.1 & 0.3 \\ 0.4 & 0.0 & -3.4 & 0.3 \\ 0.5 & -0.3 & 0.2 & -1.8 \end{bmatrix}, A_2 = \begin{bmatrix} -2.1 & 0.2 & 0.0 & 0.2 \\ 0.4 & -3.8 & 0.1 & 0.6 \\ 0.1 & 0.0 & -2.0 & 0.4 \\ 0.3 & -0.2 & 0.0 & -1.5 \end{bmatrix}, B_1 = \begin{bmatrix} 5 \\ 0 \\ -1 \\ 3 \end{bmatrix}, B_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 4 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -4.0 & 0.5 & -0.6 & 0.2 \\ 0.0 & -2.5 & 0.1 & 0.3 \\ 0.4 & 0.0 & -3.4 & 0.3 \\ 0.5 & -0.3 & 0.2 & -1.8 \end{bmatrix}, A_4 = \begin{bmatrix} -2.1 & 0.2 & 0.0 & 0.2 \\ 0.4 & -2.8 & 0.1 & 0.6 \\ 0.1 & 0.0 & -2.0 & 0.4 \\ 0.3 & -0.2 & 0.0 & -1.5 \end{bmatrix}, B_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}, B_4 = \begin{bmatrix} 5 \\ 1 \\ 0 \\ 3 \end{bmatrix},$$

$C_1 = [\,1.0\ 0.1\ 0.2\ -0.3\,], C_2 = [\,3.0\ 0.0\ 0.2\ -0.3\,], C_3 = [\,1.0\ -0.1\ 0.2\ 0.3\,],$

$C_4 = [\,2.0\ 0.1\ -1.2\ -0.3\,], D_1 = 4.5, D_2 = 1.2, D_3 = 3.3, D_4 = 2.5.$

The switching process between model is assumed to obey the semi-Markov process, and the transition probability matrix $\alpha_{ij}(h)$ meets

$$\alpha_{ij}(h) = \begin{bmatrix} (-1.4, -1.2) & (0.1,\ 0.3) & ? & ? \\ ? & ? & (0.2,\ 0.4) & (0.2,\ 0.4) \\ (0.5,\ 0.7) & ? & (-1.6, -1.4) & ? \\ (0.3,\ 0.5) & ? & ? & ? \end{bmatrix}.$$

Lemma 2.1 shows that the original system is asymptotically stable. This paper aims to solve the model reduction problem in the time-interval $[0, 4]$. Numerous simulation experiments have been conducted using MATLAB and the lower-order model is obtained.

The third-order model:

$$\hat{A}_1 = \begin{bmatrix} -2.412 & -0.024 & -0.19 \\ 0.120 & -3.872 & 0.439 \\ 0.186 & 0.148 & -4.64 \end{bmatrix}, \hat{A}_2 = \begin{bmatrix} -2.028 & -0.199 & -0.016 \\ 0.351 & -3.442 & -0.400 \\ -0.127 & -0.386 & -3.621 \end{bmatrix}, \hat{B}_1 = \begin{bmatrix} -4.354 \\ -1.162 \\ -0.953 \end{bmatrix},$$

$$\hat{A}_3 = \begin{bmatrix} -2.705 & -0.634 & 0.248 \\ -0.338 & -4.826 & 1.123 \\ 0.185 & 0.146 & -4.64 \end{bmatrix}, \hat{A}_4 = \begin{bmatrix} -2.018 & -0.181 & 0.006 \\ 0.489 & -3.208 & -0.121 \\ 0.028 & -0.125 & -3.309 \end{bmatrix}, \hat{B}_2 = \begin{bmatrix} -5.171 \\ -1.009 \\ -1.118 \end{bmatrix},$$

$$\hat{B}_3 = [\,-3.648\ -0.391\ 0.513\,]^T, \hat{B}_4 = [\,-4.736\ -2.716\ -1.742\,]^T, \hat{D}_1 = 4.888,$$

$$\hat{C}_1 = [\,-0.069\ -0.96\ 0.062\,], \hat{C}_2 = [\,-0.48\ -2.569\ -0.661\,], \hat{D}_2 = 1.934,$$

$$\hat{C}_3 = [\,-0.517\ -0.495\ 0.042\,], \hat{C}_4 = [\,0.20\ -1.175\ -1.48\,], \hat{D}_3 = 3.406, \hat{D}_4 = 3.630.$$

The second-order model:

$$\hat{A}_1 = \begin{bmatrix} -2.42 & -0.030 \\ 0.138 & -3.858 \end{bmatrix}, \hat{A}_2 = \begin{bmatrix} -2.028 & -0.197 \\ 0.365 & -3.400 \end{bmatrix}, \hat{B}_1 = \begin{bmatrix} -4.315 \\ -1.252 \end{bmatrix}, \hat{B}_2 = \begin{bmatrix} -5.175 \\ -1.132 \end{bmatrix},$$

$$\hat{A}_3 = \begin{bmatrix} -2.695 & -0.627 \\ -0.293 & -4.790 \end{bmatrix}, \hat{A}_4 = \begin{bmatrix} -2.018 & -0.181 \\ 0.488 & -3.203 \end{bmatrix}, \hat{B}_3 = \begin{bmatrix} -3.621 \\ -0.267 \end{bmatrix}, \hat{B}_4 = \begin{bmatrix} -4.740 \\ -2.653 \end{bmatrix},$$

$$\hat{C}_1 = [\,-0.067\ -0.958\,], \hat{C}_2 = [\,-0.457\ -2.498\,], \hat{D}_1 = 4.875, \hat{D}_2 = 1.738,$$

$$\hat{C}_3 = [\,-0.515\ -0.494\,], \hat{C}_4 = [\,0.188\ -1.119\,], \hat{D}_3 = 3.411, \hat{D}_4 = 4.409.$$

The first-order model:

$$\hat{A}_1 = -2.421, \hat{A}_2 = -2.049, \hat{A}_3 = -2.657, \hat{A}_4 = -2.045, \hat{B}_1 = -4.305, \hat{B}_2 = -5.110,$$

$$\hat{B}_3 = -3.586, \hat{B}_4 = -4.590, \hat{C}_1 = -0.101, \hat{C}_2 = -0.725, \hat{C}_3 = -0.485, \hat{C}_4 = 0.017,$$

$$\hat{D}_1 = 5.186, \hat{D}_2 = 2.570, \hat{D}_3 = 3.439, \hat{D}_4 = 5.336.$$

In order to estimate the performance of the proposed time-limited model reduction, we choose the outside input vector as $u(t) = e^{-t}\sin(t)$, $t \geq 0$. Numerous simulation experiments have been conducted and the results are shown in Figures 2 and 3 as below. Figure 2 described the output response of the original system (fourth order) and the reduced order model (third order, second order, and first order), which are under the same external control input. It is easy to see that the lower-order model can jump synchronously, and can approximate the original system within small error. Figure 3 depicted the reduced-order error in three cases (third order, second order, and first order). It can be seen that there exists an upper bound of the reduced-order error. The dimension of the reduced-order is lower, and the reduced-order error is greater.

To further verify the performance of the time-limited order reduction model proposed in this paper, a comparative experiment has been conducted with the balanced truncation method, and the corresponding experimental results are shown in Figures 4 and 5. Figure 4 described the output response of the original system (order 4), the reduced model based on time-limited Gramians (TL-G MR) (order 3), and the reduced model based on balanced truncation model reduction (BT MR) (order 3). It can be seen that, these two model

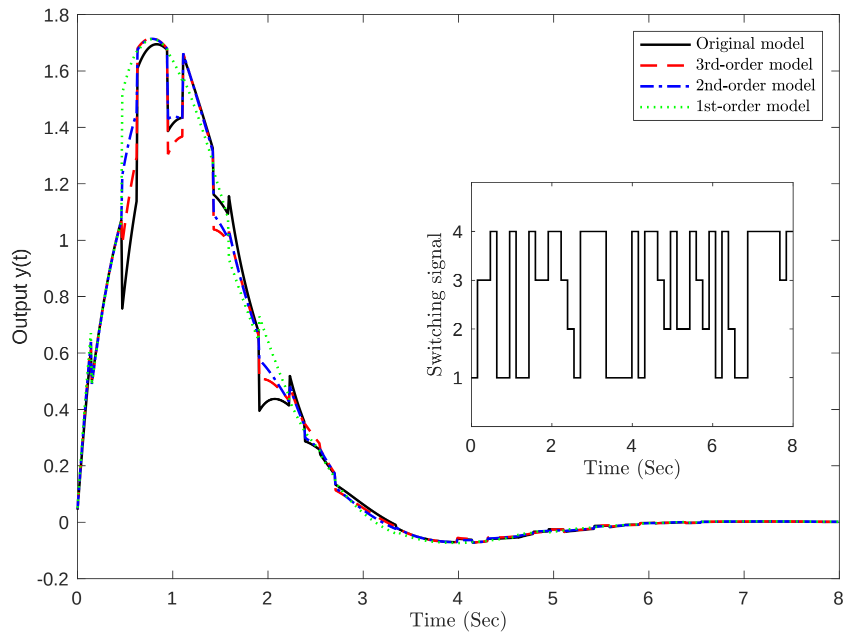

FIGURE 2. Output $y(t)$ of the original system and reduced-order models

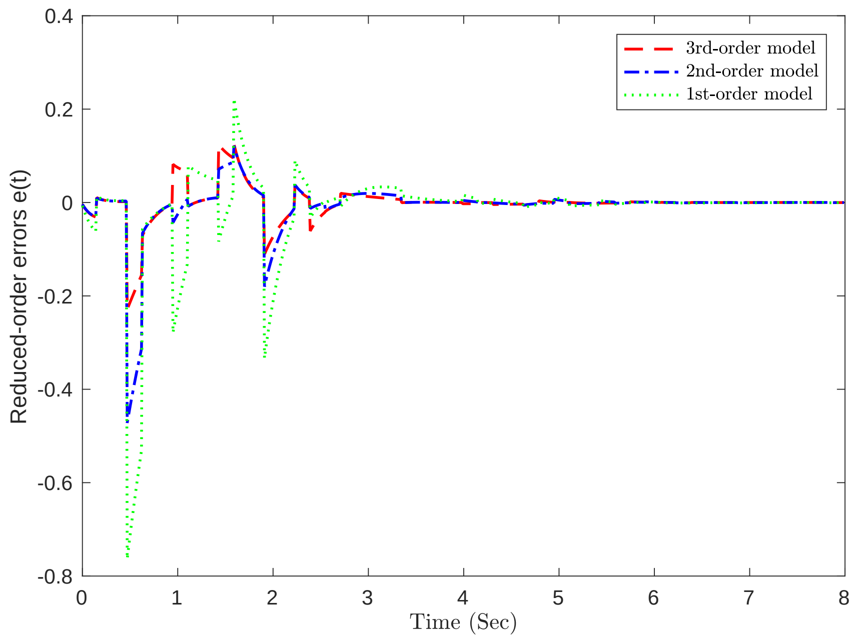

FIGURE 3. Reduced-order error (3rd, 2nd, 1st)

reduction methods both can approximate the SMJS, and can maintain the main structure and stability of the original system. Figure 5 described the reducted-order errors of these two methods. It can be seen that the time-limited model reduction method in the finite time interval $[0, 4]$ has a small error, by taking comparison experiment of these two kinds of model reduction methods. And the main structure and stability of the original system can be maintained.

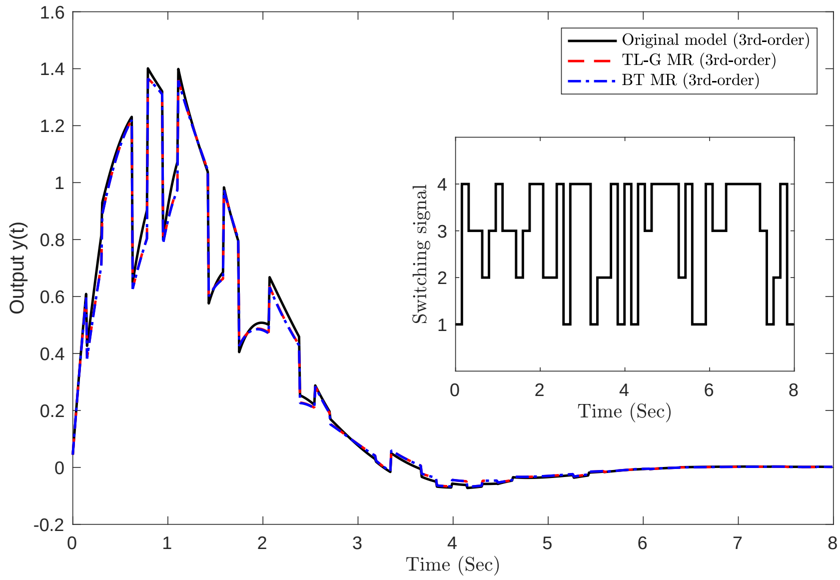

FIGURE 4. Comparison of output $y(t)$ between TL-G MR and BT MR

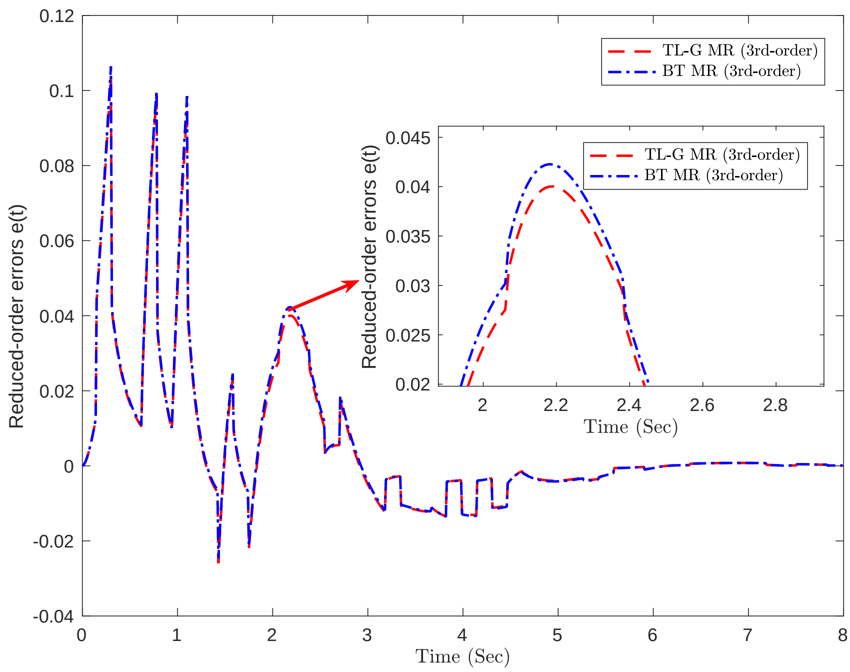

FIGURE 5. Comparison of the reduced-order error

5. **Conclusions.** This paper investigated the time-limited model reduction algorithm for a special class of continuous-time SMJS, in which the sojourn time is subject to the Weibull distribution. It is worth mentioning that, the transition probability matrix considered in this paper, which contains completely unknown and uncertain types at the same time. Specially, Lemma 2.2 was introduced to deal with the known transition probabilities. By defining the new time-interval Gramians, the solution method based on parameter-based Lyapunov equation was obtained. Then, the detailed description of

the time-limited model reduction algorithm for SMJSs was proposed in Table 1. Finally, a simulation experiment was given to show the theoretical value of the proposed model reduction method, especially compared with balanced truncation method. Furthermore, the time-limited model reduction method proposed in this paper will be adopted to handle with the time-limited reduced-order filtering for continuous-time SMJS with hybrid cyber-attacks in the future work.

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