Reflection on the Complexity of Mathematical Objects in the Initial Training of Teachers

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This work aims to know the perception of future mathematics teachers about the complexity of mathematical objects and their possible application in teaching practice, in order to improve the teaching and learning of mathematics in Basic General Education (GBS) and the Unified General Baccalaureate (BGU). For this, 19 future professors were asked about the different meanings of some mathematical objects, and they were proposed to raise contextualized problems in which a certain meaning had to be applied in their resolution. The results show that encouraging reflection on the complexity of mathematical objects in the initial training of teachers and their relationship with the design of contextualized problems affects their way of understanding mathematical competence

Keywords: initial teacher training, complexity of mathematical objects, contextualized problems

INTRODUCTION

One of the areas of study of Mathematics Education is initial teacher training, due to the important role of the mathematics teacher in the development of instructional processes. In order to improve the training of future teachers and, as a consequence, to enhance the teaching and learning processes of mathematics, teacher training courses are emphasizing the application, in the training courses, of theoretical and methodological contributions related both to the results of research in the area of Didactics of Mathematics (DM), and to some current trends concerning the teaching of mathematics (Problem Solving, use of ICT, active learning, etc.) arising from these results.

A fundamental aspect to work on in the initial training of mathematics teachers - as pointed out by different research (Font, 2007; Pino-Fan, Godino, & Font, 2011; Pino-Fan, et al, 2018; Rondero & Font, 2015) - is the reflection about the different meanings of mathematical objects and their applications in the resolution of contextualized tasks.

To a greater or lesser extent, the complexity problem associated with the mathematical object, and the link between the components in which this complexity explodes, is present in almost all the emerging theoretical frameworks in the area of Mathematics Education. In this article we will take the Onto-semiotic

Approach to Mathematical Cognition and Instruction (OSA, from now on) (Godino, Batanero, & Font, 2007 & 2019) as a theoretical reference. Working on the different meanings of a mathematical object is an aspect proposed in the OSA, where it is suggested to work on the complexity of mathematical objects through their multiple meanings.

Understanding the complexity in terms of a plurality of meanings is a result of the pragmatic view regarding meaning that is assumed in the OSA. From a pragmatic point of view, the meaning of a mathematical object is understood as the set of practices in which this object is involved in a determinant way (or not). In other words, it implies having practices regarding the field of experience that the object involves. When the meaning of a mathematical object is defined in terms of practices, as proposed in pragmatism, it appears that the meaning of a mathematical object is linked to other meanings and to other objects, since in practices this object is involved together with other mathematical objects. This fact makes it possible to make a distinction between two terms that are difficult to separate, we refer to the terms sense and meaning. Indeed, given that the object can be related to one or other objects depending on the context, the type of notation, etc. in order to give rise to different practices, we can understand sense as a partial meaning, that is, as a subset (sense) of the system of practices in which the object is determinant (meaning).

The meaning of a mathematical object understood as a system of practices can be divided into different kinds of more specific practices that are used in a specific context and with a specific type of notation, producing a specific sense. Each context helps to generate sense (allows the generation of a subset of practices), but does not generate all senses.

A mathematical object, which has been originated as an emergent of the system of practices that allows to solve a certain field of problems, with the passage of time is framed in different research programs. Each new research program allows solving new types of problems, applying new procedures, relating the object (and therefore defining) in a different way, using new representations, and so on. In this way, with the passage of time, new subsets of practices (senses) appear and extend the meaning of the object.

This work aims to know the perception of future mathematics teachers about the complexity of mathematical objects and their possible application in teaching practice, in order to improve the teaching and learning of mathematics in Basic General Education and the Unified General Baccalaureate.

The structure of the article is as follows. First, a brief explanation of some of the OSA constructs is provided, detailing the theoretical tool used in this research, followed by a description of the qualitative methodology followed, an analysis of the work of the participants in this proposal, and a discussion of the results.

THEORETICAL FRAMEWORK

In the theoretical framework we briefly explain the CCDM model of the OSA, going deeper into the representative *sample component of the mathematical object complexity to be taught* of the epistemic suitability criteria.

The CCDM Model and Didactic Suitability

The model of mathematics teacher competences and knowledge (CCDM model) is based on OSA constructs (Godino, Batanero, & Font, 2007; Godino, Batanero, & Font, 2019) and articulates several categories of mathematics teachers' knowledge and competences considered necessary for an ideal mathematics teaching (Godino, Giacomone, Batanero, & Font, 2017). This theoretical model highlights the analysis of didactic suitability, as a competence for global reflection on teaching practice, its assessment and progressive improvement (Giacomone, Godino, & Beltrán, 2018); therefore, it answers what criteria to follow in the design of task sequences to develop and assess students' mathematical competence and what changes to make in order to achieve higher learning goals. This notion is categorized into the following partial criteria of didactic suitability (Font, Planas & Godino, 2010):

✓ *Epistemic suitability*: refers to the degree of representativeness and interconnectedness of the implemented (or intended) institutional meanings in relation to a reference meaning. Tasks or

problem-situations are a critical component of this dimension, and should involve a variety of mathematical objects and processes.

✓ *Ecological suitability*: degree to which the study process is in line with the educational project of the institution, the school and society, and with the conditions of the environment in which it is developed.

✓ *Cognitive suitability*: degree to which intended and implemented meanings are in the area of potential learner development, as well as the proximity of achieved personal meanings to intended/implemented meanings.

✓ *Affective suitability*: degree of involvement (interests, emotions, attitudes and beliefs) of the students during the study process.

✓ *Interactional suitability*: degree to which didactic configurations and classroom discourse allow, on the one hand, to identify potential semiotic conflicts (that can be detected in advance), and on the other hand, to resolve conflicts that occur during the instructional process.

✓ *Means suitability*: degree of availability and adequacy of the material and time resources necessary for the development of the teaching-learning process.

According to Breda and Lima (2016) and Breda, Pino-Fan and Font (2017), a system of components and indicators is provided as a guide for the analysis and assessment of didactic suitability, which is designed for an instructional process at any educational stage.

Epistemic Suitability and the Complexity of Mathematical Objects

The components and indicators of the didactic suitability criteria have been developed taking into consideration the trends, principles and results of research in the area of Didactics of Mathematics. Particularly, for epistemic suitability, a fundamental principle of EOS has been taken into account which, with the nuances specific to each approach, is (or can be) assumed by other theoretical approaches in the area. We refer to the principle that can be stated as follows: mathematical objects emerge from practices, which entails their complexity (Font, Godino, & Gallardo, 2013; Rondero & Font, 2015). From this principle a component (representativeness) is obtained, whose objective is to take into account, as far as possible, such complexity in the design and redesign of didactic sequences (Pino-Fan, Castro, Godino, & Font, 2013).

The *Representativeness component of the mathematical object complexity* (understood as plurality of partial meanings), refers to the degree of representativeness and interconnectedness of the implemented (or intended) institutional meanings regarding a reference meaning (Giacomone, Godino, & Beltrán-Pelliecer, 2018). Each of these meanings allows solving different types of problems, so if you want to apply the mathematical object to solve different problems (mathematical competence) it is necessary to teach a representative sample of partial meanings (Font, Breda, & Seckel, 2017).

The following table (Font, Breda, & Seckel, 2017), explains in detail, the indicators of the *Representativeness* component of the epistemic suitability criteria.

TABLE 1
THE REPRESENTATIVENESS COMPONENT AND ITS INDICATORS

Epistemic Suitability Component	Indicators
Representativeness	✓ Partial meanings (definitions, properties, procedures, etc.) are a representative sample of the mathematical notion's complexity to be taught. ✓ Partial meanings (definitions, properties, procedures, etc.) are a representative sample of the complexity contemplated in the curriculum of the mathematical notion to be taught. ✓ For one or more partial meanings selected for implementation, is a representative sample of problems considered? ✓ For one or several partial meanings selected for implementation, is the use of different modes of expression (verbal, graphic, symbolic...), treatments and conversions between them considered?

Source: Font, Breda, and Seckel (2017).

Investigations Regarding the Complexity of Different Mathematical Objects

Different researches have been carried out in order to deepen the complexity of different mathematical objects: natural numbers (Godino, Font, Wilhelmi, & Arrieche, 2009), arithmetic mean (Rondero & Font, 2015), limit (Contreras, García, & Font, 2012), optimization (Balcaza, Contreras, & Font, 2017), proportionality (Godino, Beltrán-Pellicer, Burgos, & Giacomone, 2017; Burgos, Castillo, Beltrán-Pellicer & Godino, 2020), Tales' Theorem (Font, Breda, & Seckel, 2017), derivative and antiderivative, as well as students' understanding of such complexity (Pino-Fan, Godino, & Font, 2011; Pino-Fan, Castro, Godino, & Font, 2013; Pino-Fan, Font, Gordillo, Larios, & Breda, 2018; Pino-Fan, Godino, & Font, 2018), inequation (Monje, Seckel, & Breda, 2018).

For the mathematical object *"derivative"*, Pino-Fan, Godino and Font (2011) characterize the complexity of the mathematical object by nine primary object configurations: 1) tangent in Greek mathematics; 2) variation in the Middle Ages; 3) algebraic methods for finding tangents; 4) kinematic conceptions for tracing tangents; 5) intuitive ideas of limit for calculating maxima and minima; 6) infinitesimal methods in calculating tangents; 7) Methodus fluxionum; 8) calculation of differences; and, 9) derivative as limit. In Pino-Fan, Castro, Godino and Font (2013) these nine configurations are used for the reconstruction of the global meaning of the derivative, which is used to assess the representativeness of the intended meaning in the Mexican high school curriculum (from the configurations of primary objects activated in the mathematical practices proposed both in the Curriculum and in the textbooks of that level).

The complexity characterization of the derivative made in Pino-Fan, Godino and Font (2011) facilitates having elements to design questionnaires to identify the understanding of students, future teachers or in-service teachers about the derivative. In Pino-Fan, Godino and Font and (2018) a questionnaire was designed in order to determine prospective teachers' understanding about the derivative in which tasks that activate the different partial meanings of the derivative characterized in Pino-Fan, Godino and Font (2011) were included.

In Gordillo and Pino-Fan (2016) the antiderivative complexity is defined by four primary object configurations related to four basic problems: a) the geometric problem of the tangents of a curve and the quadrature of the curve; b) the problem of the relation fluxions - fluents; c) the problem concerning the relation of differentials and summations; and d) the problem of the identification of elementary functions. The characterization of such complexity allows us to have elements available to design questionnaires that enable us to identify the understanding of students, future teachers or in-service teachers about the antiderivative. In Gordillo, Pino-Fan, Font, and Ponce (2018) and in Pino-Fan, Font, Gordillo, Larios, and

Breda (2018) a questionnaire was designed to determine university students' understanding about the antiderivative in which tasks activating the different partial meanings of the antiderivative characterized in Gordillo and Pino (2016) were included.

Monje, Seckel and Breda (2018), by means of a comparative analysis between the complexity of the mathematical object, inequation with the national curriculum and the school texts granted by the Ministry of Education of Chile, concluded that the treatment given to the mathematical object under study (inequalities) does not consider all the necessary components for the teaching of the inequation from its complexity, in particular, it was observed that both the curriculum and the school texts leave out, in particular, quadratic inequalities and inequalities with absolute value.

These investigations concluded that teachers should take into account the complexity of the mathematical objects they teach in order to obtain more effective teaching, which led the authors of this article to become interested in how to incorporate the problem of the mathematical object complexity in teacher training.

OBJECTIVE

Understanding the perception that future mathematics teachers have about the importance of considering the complexity of mathematical objects in their teaching practice.

METHODOLOGY

In this section, we provide the study context (institution involved, research participants, etc.) and the qualitative methodology used to analyze the participants' responses.

Study Context

Nineteen future mathematics teachers who are studying the second semester of the Advanced Algebra course of the Experimental Sciences Pedagogy Career: Mathematics and Physics, at the University of Cuenca, participated in the study.

Research Study Phases

In the first phase, the students jointly with the teacher agreed on a series of mathematical objects, of which they would deepen about their complexity. In the second phase, with the objects assigned and the groups organized, they were provided with literature related to the object on which they had to reflect. In a third phase, the group had to ask itself questions such as what are the partial meanings of the mathematical object worked on by the students and what representations could be given to the partial meanings of these mathematical objects. In a fourth phase, they were also asked to propose problems for each meaning. In the fifth phase, each group reported its reflections to the entire group. In a sixth phase, the whole group solved the tasks that had been proposed in each small group. Finally, the students discussed the results obtained, analyzing their application and generalization, through conclusions and recommendations to take into account the mathematical objects complexity in the instructional processes.

RESULTS

The students participating in this study were organized into five groups of four or five students, sharing their proposals through a debate of ideas and presenting the following information as a result:

TABLE 2
WORK DONE BY STUDENTS PARTICIPATING IN THE PROPOSAL. GROUP NO. 1

MATHEMATICAL OBJECT ADDRESSED	Group No. 1: Complex numbers
ANALYSIS **OF THE** **REPRESENTATIVENESS** **AND THE** **ACTIVITY** **PRESENTED**	The partial meanings are a representative sample of the complexity of the mathematical notion to be taught, and they are also included in the curriculum:

Meaning 1: Algebraic.	The unit of imaginary numbers is √ -1 and is represented by the letter i, formed by a real number and an Imaginary one.
Meaning 2: Geometric	This is done using a system of rectangular or Cartesian axes, where the real numbers are represented on the "x" axis and the imaginary quantities on the "y" axis. The plane formed by the real and imaginary axes is called Argand diagram.
Meaning 3: Trigonometric	When we have a **complex number in polar form** (therefore, it is defined by just giving [Z] and "a") we can easily pass to the **trigonometric form** or also called argument module; that is, the complex number is given by its module and its angle.

For one or more partial meanings selected for implementation, a representative sample of problems is considered:

Meaning 1: Algebraic.	The quotient of two complex numbers is pure imaginary, their sum is real and is equal to 5. The number of the dividend is twice the number of the divisor. Finding the divisor
Meaning 2: Geometric	Given the complex numbers, express in Argand diagram $Z_1 = 3 + 3i$ $Z_2 = 1 - 2i$ $Z_3 = -2 - 2i$ $Z_4 = -3 + i$
Meaning 3: Trigonometric	Convert the following complex number to its polar form: Z = 1 - i Sum of complexes in binomial form

In addition to the use of different ways of expression, treatments and conversions between them:

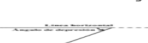| Meaning: Algebraic.	In the complex number a+ bi, "a" is called the real part and "bi" the imaginary part. When a = O, the complex number is called pure imaginary. If b =0 the complex number reduces to the real number "a".
Meaning 2: Geometric	By means of the vectorial sum we can obtain the geometrical shape representation
Meaning 3: Trigonometric	To represent a complex number in this way, it is necessary to know the "radio vector", known as "module" and the angle that it forms with the positive part of the "x" axis.

Source: developed by the authors.

Analyzing the activity performed by group No. 1, it can be evidenced that they comply with the *representativeness* of the epistemic suitability criteria, through their three partial meanings: algebraic, geometric and trigonometric; in addition to a representative sample of problems and the use of different ways of expression, such as verbal, graphic, symbolic and others.

After the discussion of this work, the presenters applied the following task, obtaining as a result that, out of the 15 participating students, 12 answered correctly to the questions posed in the task and only three students made minimal errors, as stated by the evaluators of group 1: "They correctly identified the algebraic meaning, but had a small error because they did not distinguish the properties of complex numbers".

FIGURE 1
TEST APPLIED BY GROUP NO. 1.

> **University of Cuenca**
>
> Pedagogy of Experimental Sciences
>
> **Complex Numbers**
>
> Name:…………………………………………………………..   Date: 07/01/2019
>
> **1 Calculate E =** $\dfrac{x + y}{X - y^3}$
>
> If it is fulfilled that $(1 + i)2 + (1 + i)4 + (1 + i)6 + (1 + i)8 = x + yi$
>
> What is the meaning of complex number that you have used?
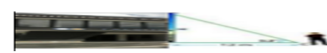
Source: Developed by the Authors

The activity performed by the following group is detailed below:

TABLE 3
WORK DONE BY STUDENTS PARTICIPATING IN THE PROPOSAL. GROUP NO. 2

MATHEMATICAL OBJECT ADDRESSED	*Group No. 2: Tales' Theorem*
ANALYSIS OF THE REPRESENTATIVENESS AND THE ACTIVITY PRESENTED	The partial meanings are a representative sample of the complexity of the mathematical notion to be taught, and they are also included in the curriculum: Meaning 1: **Geometric Meaning** — Parallel lines. Similar triangles. https://www.academia.edu/7659282Calvache_G_-_Plane and Space Geometry PDF Meaning 2: **Algebraic Meaning** — Proportionality - Ratio. https://es.calameo.com/read/003671470afdd13dea2c2 Meaning 3: **Trigonometric Meaning** — Trigonometric Functions. https://proyectodescartes.org/iCartesiLibri/materiales_didacticos/trigonometria/c1p1.html Meaning 4: **Arithmetic Meaning** — Rule of three. https://www.smartick.es/blog/matematicas/algebra/regla-de-3-simple/
	For one or more partial meanings selected for implementation, a representative sample of problems is considered:

In addition to the use of different ways of expression, treatments and conversions between them:

Meaning 1:	In a triangle ABC we trace a line parallel to the side BC from a point B such that AB is 0.25 A. What is the similarity ratio?
	Draw a right angle triangle with sides 15 and 8 cm. If its midpoints are joined, does it result in a triangle similar to it?
Meaning 2:	Nine people perform a job in 16 days. How long will it take 8 people to perform the same job?
	Upon arrival at the hotel we were given a map with the city's interesting places, and we were told that 5 centimeters of the map represented 600 meters of reality. Today we want to go to a park that is 8 centimeters away from the hotel on the map. How far is this park from the hotel?
Meaning 3:	An electrician, standing on a pole, observes his assistant on the ground 25 meters from the foot of the pole with a depression angle of 40°. Calculate the post height.
	From a point on the ground, a student in algebra class observes the highest part of the philosophy faculty building with an elevation angle of 53° when he is 12 meters away from its base. What is the height of the faculty building?
Meaning 4:	Sergio is in a photo with his friend Enrique. In the photo Sergio measures 4.5 cm and Enrique measures 4.25 cm. If Enrique is 1.7 meters tall in reality, how tall is Sergio?
	Measure on the map the distance between the cities: Cuenca - Guayaquil and Quito - Portoviejo, find out what are the real distances between these cities. Scale: 1:50
	Distance Cuenca - Guayaquil: 4 cm. Distance Quito - Portoviejo: 7 cm

	Geometric Meaning
Meaning 1:	Parallel lines: The segments of two transversal lines intercepted between parallels are proportional.
	Similar triangles
	• The corresponding angles must be congruent.
	• The corresponding sides must be proportional.
	If the corresponding angles are congruent
	And the corresponding sides are proportional. Then we say that the equivalence is a similarity, and it is written: $\triangle ABC = \triangle A'B'C'$
	The ratio of any two corresponding sides is the similarity ratio. It should be emphasized that this equivalence is not a congruence because to be a congruence the similarity ratio should be equal to unity.

	Algebraic Meaning
Meaning 2:	**Proportionality:**
	Proportionality is a constant relationship or ratio between different measurable quantities. If one increases or decreases the other also increases or decreases proportionally, i.e. when two quantities vary directly with respect to each other.
	Ratio between two numbers: Whenever we talk about Ratio between two numbers we are referring to the quotient (the result of dividing them) between them. So:
	Ratio between two numbers "a" and "b" is the quotient between them: a / b
	For example the Ratio between 10 and 2 is 5, since 10 / 5 = 2

	TRIGONOMETRIC MEANING
Meaning 3	**TRIGONOMETRIC RATIOS**
	Trigonometric ratios allow us to relate the magnitudes of angles and sides of a triangle. It is important to define them in a right angle triangle in order to be able to use the Pythagorean Theorem to obtain magnitudes of unknown sides and to find the remaining angles different from 90°.
	Elevation angle: it is the angle between the horizontal plane and the line of sight to the object.
	Depression angle: It denotes the angle from the horizontal to the bottom of an object.

	ARITHMETIC MEANING
Meaning 4	Rule of three: is a mechanism that allows the solution of problems related to the similarity between values, of which three are known and the fourth is an unknown. Thanks to this rule, the value of this fourth term can be discovered. In the simple rule of three, a similarity relation is established between two known values A and B and knowing a third value C, a fourth value D is calculated.
	$D = \dfrac{C \cdot B}{A}$
	The similarity ratio between A and B must necessarily be direct when, within this proportionality, a higher value of A also corresponds to a higher value of B (or a lower value of A corresponds to a lower value of B).

Source: developed by the authors.

Similar to the previous case, it can be seen that the group No. 2, complies with the representativeness component of the epistemic suitability criteria, where it is expressed in four partial meanings: algebraic, arithmetic, geometric and trigonometric.

When applying the evaluation, they obtain as a result that, out of the 14 participating students, seven demonstrate a good application and connection of meanings with the problems posed, while the others, having an error, in the words of the group: "they recognize and apply the connection of the meanings with the problems posed".

FIGURE 2
TEST APPLIED BY GROUP NO. 2

Source: Developed by the Authors

The detail of the next group's activity is:

TABLE 4
WORK DONE BY STUDENTS PARTICIPATING IN THE PROPOSAL. GROUP NO. 3

MATHEMATICAL OBJECT ADDRESSED	Group No. 3: Equations
	The partial meanings are a representative sample of the complexity of the mathematical notion to be taught, and they are also included in the curriculum:

ANALYSIS OF THE REPRESENTATIVENESS AND THE ACTIVITY PRESENTED	For one or more partial meanings selected for implementation, a representative sample of problems is considered:
	<table><tr><td>Meaning 1:</td><td>Find the solution set of the equation: 4[3x – (5x – 1)] = 3 – 4x</td></tr><tr><td>Meaning 2:</td><td>Find the equation of the line that passes through the point A(2, -4) and has a slope of (-1/3).</td></tr><tr><td>Meaning 3:</td><td>Solve the following equation and give the results in degrees. Sen x = 1</td></tr></table> In addition to the use of different ways of expression, treatments and conversions between them: <table><tr><td>Meaning 1:</td><td>E = F</td></tr><tr><td>Meaning 2:</td><td>Y – y₁ = m (x – x₁)</td></tr><tr><td>Meaning 3:</td><td>Tan²x + Csc²x – 3 = 0</td></tr></table>

Source: developed by the authors.

Group No. 3 works on the representativeness of the mathematical object "equations" in a very elementary way and analyzes the evaluation of six of their classmates, where four of them answered the evaluation correctly and only two could not relate the meanings with the proposed problems.

It can be concluded that, in spite of presenting a rather simple work, the presenters' understanding regarding the mathematical object "equations" complexity is visible.

FIGURE 3
TEST APPLIED BY GROUP NO. 3.

Problem 1: Find all the solutions of the following equation:

Sen x + cos x = √2
a. Relate it to a type of meaning.
b. Write down why you related to with that meaning.

Problem 2: "Passerby, this is the tomb of Diophantus: this is the one who with this surprising distribution tells you the number of years he lived. His childhood occupied the sixth part of his life; then during the twelfth part his cheek was covered with the first face hair He spent yet another seventh part of his life before he married, and five years later, he had a precious child who, having reached half his father's age, perished of an unfortunate death. His father had to survive him, mourning him, for four years. From all this we can deduce his age".

Relate to one type of meaning.

Source: Developed by the Authors

The following group presents its work, according to the following details:

TABLE 5
WORK DONE BY STUDENTS PARTICIPATING IN THE PROPOSAL. GROUP NO. 4

MATHEMATICAL OBJECT ADDRESSED	Group No. 4: Fractions
ANALYSIS OF THE REPRESENTATIVENESS AND THE ACTIVITY PRESENTED	The partial meanings are a representative sample of the complexity of the mathematical notion to be taught, and they are also included in the curriculum: Meaning 1: ALGEBRAIC Meaning 2: ARITHMETIC Meaning 3: GEOMETRIC For one or more partial meanings selected for implementation, a representative sample of problems is considered: Meaning 1: A soccer field has unknown measurements. Nevertheless, a maintenance operator tells us that the ratio between width and length less twenty meters is equal to one half. Likewise, the sum of the length and width is 170 meters. What are the measurements of the soccer field? Meaning 2: Ester has spent 1/3 of her grandparents' allowance money to buy an adventure book. She also spent 1/9th of her money on buying a bag. What fraction of her money has Esther spent? Meaning 3: At a birthday party there are 4 yellow balloons, 2 orange balloons, 1 blue balloon and 5 red balloons. What fraction represents each color? If 5 are burst, we still have 7 balloons left, what fraction of the total are left without bursting? In addition to the use of different ways of expression, treatments and conversions between them: Meaning 1: An expression that has at least one letter in the numerator or denominator. Meaning 2: It is the quotient of two numbers where the denominator must be different from zero, and the numerator must be integer. Meaning 3: It is considered as a whole that is divided into equal parts essentially indicating the relationship between the whole and a designated number of parts.

Source: developed by the authors.

Group No. 4 shows compliance with the representativeness component, through algebraic, geometric and trigonometric meaning. On the other hand, in the evaluation applied to 15 students, 11 of them "Apply correctly the algebraic, geometric or trigonometric meanings, as appropriate"; while the others do not justify their given answers.

The next group shares its proposal in the following way:

FIGURE 4
TEST APPLIED BY GROUP NO. 4

4. EVALUATION OF THE PROPOSAL
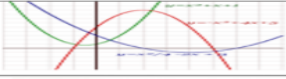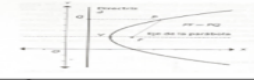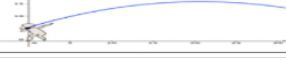
The following table shows the results obtained from the evaluation applied to our classmates, the purpose of which is to demonstrate whether the understanding of the meanings of the mathematical object Fractions is achieved.

Problem 1: Solve:	Problem 2: Determine to which meaning it belongs. Please justify your answer.
$E = \dfrac{a-b}{(b+c-a)(b-c-a)} + \dfrac{b-c}{(c+a-b)(c-a-b)} + \dfrac{c-a}{(a+b-c)(a-b-c)}$	

Source: Developed by the Authors

TABLE 6
WORK DONE BY STUDENTS PARTICIPATING IN THE PROPOSAL. GROUP NO. 5

MATHEMATICAL OBJECT ADDRESSED	*Group No. 5: Quadratic Functions*
ANALYSIS OF THE REPRESENTATIVENESS AND THE ACTIVITY PRESENTED	The partial meanings are a representative sample of the complexity of the mathematical notion to be taught, and they are also included in the curriculum: Meaning 1: ALGEBRAIC — They are polynomial functions of second degree of the form $f(x) = ax^2 + bx + c$ Meaning 2: GEOMETRIC — A parabola is a conic section which can be represented as follows. $F(x) = a(x - h)^2 + k$ Meaning 3: PHYSICAL — In the path followed by objects thrown upward and at a certain angle, the parabola represents the path of the ball (or rock, or arrow, or whatever has been thrown). $\Delta h = Vo + \frac{1}{2}gt^2$
	For one or more partial meanings selected for implementation, a representative sample of problems is considered: Meaning 1: ALGEBRAIC — Consider the function "g" of equation $g(x) = x^2 + bx + c$. Knowing that $g(-1) = -6$ and $g(2) = -12$, write the equation of the function g and plot it. Meaning 2: GEOMETRIC — Find the length of the focal chord of the parabola $x^2 + 8y = 0$ which is parallel to the line $3x + 4y - 7 = 0$. Meaning 3: PHYSICAL — A fire hose shoots water at an angle of 48. How fast must the water come out so that the stream reaches a window 18 m above the ground if the firefighter is 20 m from the base of the building?

	In addition to the use of different ways of expression, treatments and conversions between them:

Source: developed by the authors.

Problem D

Given the equation $X^2 + 6x + ky = 0$
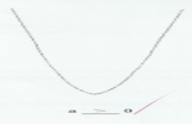
For what value of K does it represent a parabola whose vertex belongs to the line $y=6$? For the value of K found, indicate vertex, focus and directrix.

Geometric Meaning

2. From what you have seen in class, select true or false. (2P)   (2)

- A parabola will always have an intersection on the y-axis (ordinate)   T
- If the discriminant is less than zero, the parabola has its vertex at a point on the abscissa axis(x)   F
- The domain of a quadratic function is not constant   F
- A parabola is concave upward if the value of a is equal to 0   F

3. Complete the formulas with the sign <, >, =, according to the   (1.5) graph (1,5P).

Analyzing the activity performed by group No. 5, it can be observed that representativeness is taken into account through algebraic, geometric and physical meanings; and in the evaluation applied to 14 classmates, eight have an "excellent performance", four have a "normal performance" and only two, a "low performance" (as indicated by the presenters).

FIGURE 5
TEST APPLIED BY GROUP NO. 5

UNIVERSITY OF CUENCA
FACULTY OF PHILOSOPHY, LITERATURE AND EDUCATIONAL SCIENCES
EXPERIMENTAL SCIENCE PEDAGOGY DEGREE PROGRAM
QUADRATIC FUNCTIONS LESSON

NAME: Marlon Rilovo P
SUBJECT: Advanced Algebra   GROUP: I   CYCLE: 2nd
TEACHERS: Sebastián Calero, Tania Orellana, Joseline Peralta and Cristina Tenesaca.
DATE: 2019/07/04
Use pencil or mechanical pencil to solve the exercises, mark the answers with a circle using blue ink pen, the use of liquid paper is allowed.

1- From what was exposed in the QUADRATIC FUNCTIONS class, relate the following problems with the meanings exposed in the class. (2p)

Problems:
Problem A: The profit motive (in thousands of dollars) of a company is given by:

$P(x) = 5000 + 1000x - 5x^2$

Where X is the amount (in thousands of dollars) the company spends on advertising. Find the quantity x, which the company has to spend to maximize its profit.

Algebraic Meaning

Problem B: An object is launched vertically upward with an initial velocity of Vo ft/sec. Its distance S (t), in feet above the ground is given by:

S (t) - -16t² - V₀t

Search V₀ so that the highest point the object can reach is 300 feet above the ground.

Physical meaning

Problem C: Plot and state the properties of the function f whose equation is $f(x) = x^2 - 4x - 5$.

Algebraic meaning

Source: Developed by the Authors

RESULTS DISCUSSION

Of the 19 students who participated in this activity, the majority agreed that mathematical objects can be approached from their different meanings, as shown in Figure 6:

FIGURE 6
STUDENT'S RESPONSES

> **6. Do you think that mathematical objects can be represented in different ways?**
>
> **(2 P)**
> **Explain why.** Yes, because as we explained above there are several ways to explain these objects in such a way that the information is provided in a variety of ways so that it reaches all our students.

> **8. Nowadays it is recommended to develop mathematical competencies; that is, students are expected to be able to solve different problems by applying the content that has been taught. Do you think that if we had explained only one of the proposed meanings, about arithmetic mean, the student would have been able to solve all the proposed problems? If your answer is no, what do you suggest in order to solve all the problems? (4P)**
> No, because some may understand and others may not, so it is suggested to address various meanings with known words for greater student comprehension. In addition, all the characteristics of the topic should be gathered in order to make the subject matter even better known.

Source: Developed by the Authors

Once these proposals of multiple meanings of mathematical objects and their respective evaluations have been concluded and shared, the perception of these future teachers is presented through the *conclusions and recommendations* of each of the groups:

TABLE 7
CONCLUSIONS AND RECOMMENDATIONS ISSUED BY THE STUDENTS
PARTICIPATING IN THE PROPOSAL

WORKING GROUPS	Conclusions and recommendations
Group **No. 1**	The results of this work have been successful since a single topic can be broken down into smaller but no less important ones, that means, the topic is related to more branches of mathematics which can be useful to relate it and that students integrate the meanings, it is a very good proposal since students know more than a single meaning and they can work in their own way, so that they can reach the same result without having applied the same process, I would recommend to continue working with this type of proposals since the results are good.
Group **No. 2**	Upon observing and analyzing the results of the work, it is concluded that the application of this methodology within the classroom has been successful, since most of our classmates obtained "correct" in the majority of the problems posed. It can be stated that, in order to achieve meaningful learning with successful learning achievements, it is necessary to introduce different meanings of mathematical objects, since each student has his or her own learning and assimilation style. Concluding the peer review, it is clear that each one adopted the best meaning in accordance with the context of the problem, in order to facilitate the resolution of the problems. In our opinion, we consider that the use of didactic material for the explanation of the different meanings would be a good way to achieve meaningful learning in students.

	From our point of view, we think that applying exercises that illustrate each of the meanings of mathematical objects would help to achieve successful results. Before concluding, we would like to suggest a last recommendation based on the results and conclusions reached after the evaluation and research, so we can say that, if teachers apply this methodology in the classroom, they will achieve significant learning in students, since each one will choose the meaning that best suits their understanding or the context of the problem, obtaining excellent results.
Group **No. 3**	As a conclusion, we can apply this proposal since the subject is one of the most basic within algebra and it is also the basis for the next years. In the same way, students can develop their capabilities. In addition, it is a didactic proposal in which students can interact with the teacher and their classmates, thus keeping a satisfactory environment. Furthermore, we can use didactic materials and apply learning methodologies such as constructivism and technology by using applications such as GeoGebra. However, many times there will be conditions in which students, even though they know the required mathematical meanings, tend to confuse some things such as the case of trigonometric identities or factorization.
Group **No. 4**	Fractions are indispensable to our lives, whether in a job or in educational institutions. In order to apply them, it is necessary to have an understanding of the concept. It is very important to clarify the meanings of mathematical objects in order to facilitate the learning of mathematics. The use of didactic material is essential for facilitating understanding in the teaching-learning process; it also helps teachers and future teachers who experience this way of working to reinforce the contents they already have about fractions and to significantly build methodological strategies that will be useful for conceptualizing the notion of fractions. As a conclusion, when applying the test, the results we obtained were very positive, since the great majority of the students know how to recognize and interpret mathematical meanings, so we can say that there was significant learning. Students should know the properties to be applied when solving operations with fractions. Making the meanings of different mathematical objects known is a strategy that can really contribute a lot to teaching practice and facilitate student learning.
Group **No. 5**	As a consequence of the constant technological evolution, nowadays there are different sources in which we can find a lot of information related to the topic we need, either through web pages or virtual libraries, which are very useful tools for both students and teachers in order to keep up with the constant innovation of knowledge. There are different ways to teach a certain subject either in the area of mathematics or physics, mathematical objects have several meanings which are very helpful for teachers because they can choose the meaning that best suits the needs of students and the topic that needs to be explained depending on the subject they are teaching, so they can explain correctly and can reach their students so that they learn in a meaningful way and do not have any complications in higher courses. In order to confirm that the students are learning, the teacher must apply some periodic evaluations on the topics that have been previously explained, to which a grade is assigned to reflect the level of knowledge that the student has about the subject. We should not abuse the use of technology because it can be a great distraction, for this reason there are students who use it excessively and many times they do not use it for academic purposes. Both teachers and students should make sure that the sources from which they obtain information are reliable, since there are several sources in which the information is false.

Source: developed by the authors.

Therefore, it is concluded that, after this instructional process, the participants had the perception that it is important to take into account the complexity of the mathematical objects to be taught in the teaching practice, in order to achieve significant learning in the students of Basic General Education and High School. They also state that the different meanings should be presented gradually.

It is also highlighted in the responses that the participants showed different meanings of mathematical objects, both intra, as well as extra-mathematical, such is the case of group No. 5 that considered the meanings within Physics that also corresponds to their professional training (Teaching in Mathematics and Physics); which can be considered as evidence of their ability to work, not only in intra-mathematical contexts, but also in extra-mathematical contexts, i.e. as a connection that can be given between mathematics and the real world (Font, et al, 2017).

CONCLUSIONS AND IMPLICATIONS

In different training processes for mathematics teachers in Spain, Mexico, Brazil, Ecuador, Chile, Panama, Costa Rica, Venezuela and Argentina, a series of investigations have been carried out to investigate the use of the didactic suitability criteria construct as a tool to organize the teacher's reflection on his/her practice, when this reflection is focused on the evaluation and improvement of the practice, with the objective of developing in teachers the subcompetence of analysis of the didactic suitability of an instructional process. These investigations have revealed the following aspects: 1) although the teaching of the components and indicators of the didactic suitability criteria are not explicitly incorporated, some of them, and in particular the component <<representative sample of the mathematical object complexity>>, are implicitly present when teachers or future teachers make evaluations of didactic proposals (their own or others') (Breda, 2020; Breda and Lima, 2016; Breda, Pino-Fan and Font, 2017). 2) Incorporating the component <<representative sample of the mathematical object complexity>> to assess the epistemic suitability of a teaching and learning process, is not an easy task, neither for trainers nor for their students (future teachers or in-service teachers), but it can be taught as part of the teacher training process. In these training devices, emphasis is placed on the need to carry out a preliminary study focused on the reconstruction of a global meaning of the mathematical object to be taught in order to be aware of its complexity.

According to the above mentioned research, as a result of the experience, students in teacher training are aware that, in order to improve the learning of mathematics, it is necessary to take into consideration the complexity of mathematical objects, understood as the different ways of understanding the meanings and their application in the resolution of problems. In addition, analyzing what the participants worked on in this proposal, it gives us the idea that they could progress in their reflection and deepen in the complexity articulation associated to the mathematical object as a previous and necessary step to advance to a unitary vision of the mathematical object (Rondero and Font, 2015).

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